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A note on the laminar mixing of two uniform parallel semi-infinite streams

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According to classical boundary-layer theory, when two uniform parallel streams are brought into contact at large Reynolds number (R) the location of the dividing streamline remains indeterminate to $O(R^{-\frac{1}{2}})$ if both streams are subsonic and semi-infinite in extent. It is demonstrated here that this indeterminacy is a fundamental property of such a system which cannot be resolved, as Ting (1959) proposed, by balancing the pressure across the viscous mixing region to higher order in R.

1. Introduction

It is well known that in the laminar mixing of two uniform parallel streams at large Reynolds number (R) a thin viscous boundary layer forms between them downstream of their point of contact. Because this mixing region is of only $O(R^{-\frac{1}{2}})$ in thickness, it is natural to assume when analysing its structure that the parallel streams are effectively semi-infinite in extent. Consequently, the problem of mixing between two uniform semi-infinite streams has received considerable attention over the years, since it is believed to model a variety of mixing phenomena.

It was established some time ago that, in the viscous mixing region referred to above, the appropriate laminar boundary-layer equations admit a Blasius-type similarity solution which has been studied in detail by numerous investigators (Görtler 1942; Keulegan 1944; Lessen 1949; Lock 1951; Crane 1957). This solution is not unique, however, because, although the similarity form of the boundary-layer equation is of third order, the only apparent boundary conditions are that the longitudinal component of the velocity at the upper and at the lower edge of the boundary layer should match the velocity of the corresponding uniform stream. Moreover, the additional boundary condition required for a unique solution cannot be obtained for this problem by the usual procedure of balancing the $O(R^{-\frac{1}{2}})$ component of the pressure across the boundary layer since this condition is identically satisfied when both streams are subsonic. As a result, an indeterminacy which corresponds to an $O(R^{-\frac{1}{2}})$ displacement of the streamline ($\Psi = 0$) separating the two streams arises in the solution.

To resolve this non-uniqueness, Ting (1959) derived a compatibility relation to serve as the third boundary condition by carrying out the analysis to higher

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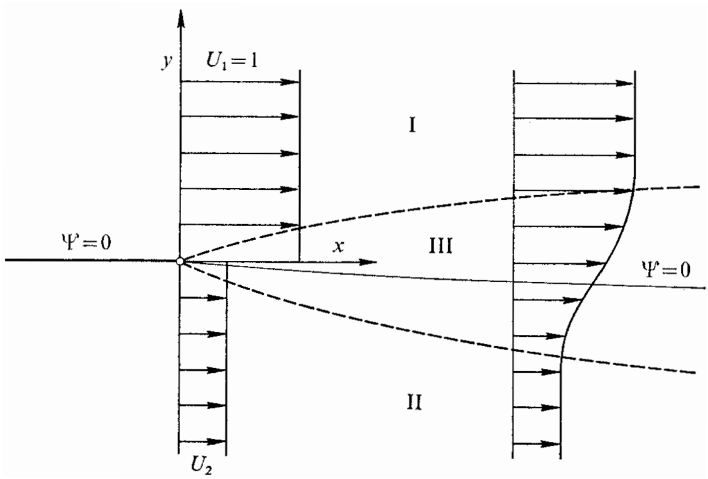

FIGURE 1. Mixing of two uniform semi-infinite streams.

order in R and matching the pressure to $O(R^{-1})$ across the boundary layer. In performing this higher order analysis, however, Ting, did not consider higher order corrections to the position of the $\Psi = 0$ streamline, with the result that his compatibility condition is incomplete. For this reason, and also because Ting's result has been referred to in a number of studies dealing with various mixing problems (Ting & Ruger 1967; Casarella & Choo 1968; Mills 1968; Viviand 1969), we shall here re-examine in detail the higher order solution to the relevant equations for the case of two semi-infinite uniform incompressible streams having identical physical properties, and shall demonstrate that when all higher order effects are included the position of the dividing streamline remains indeterminate. Specifically, we shall show that balancing the pressure across the boundary layer to higher order does not allow the $O(R^{-\frac{1}{2}})$ displacement of the dividing streamline to be calculated since unknown constants which correspond to additional higher order effects appear in the appropriate expression for this pressure balance.

2. Re-derivation of Ting's compatibility condition

In analysing this laminar mixing problem, variables are rendered dimensionless in the usual manner using the uniform velocity U_1' of the faster moving stream and a characteristic length l, which may be chosen arbitrarily owing to the absence of a length scale in the problem. The Reynolds number is then given by $R = U_1'l/\nu$, where ν is the kinematic viscosity of the fluid. Also, the velocities of the two uniform streams become unity and U_2, respectively, with $U_2 < 1$.

The basic features of the flow for this case are, of course, well known. As depicted in figure 1, two irrotational inviscid streams I and II, of essentially uniform velocity, are separated by a viscous boundary layer III, which is of $O(R^{-\frac{1}{2}})$ in thickness. To first order, the flow in this boundary layer is described by Lock's (1951) solution, except that the displacement of the $\Psi = 0$ streamline from the $+x$ axis must now be included in the similarity variable η. Since the resulting solution cannot depend on the choice of length scale l, the position of the dividing streamline $y = \phi(x; R)$ is given by

$$\phi(x;R) = (x/R)^{\frac{1}{2}}\{\alpha_1+\alpha_2(Rx)^{-\frac{1}{2}}+\ldots\} \tag{2.1}$$

and thus, in order that $\eta = 0$ along the $\Psi = 0$ streamline,

$$\eta = (R/x)^{\frac{1}{2}}\,[y-\phi(x;\,R)]. \tag{2.2}$$

In (2.1), the α_n's are unknown constants which must be evaluated if the solution is to be unique. This correction to Lock's (1951) solution corresponds to that presented by Ting (1959), except that he considered only the first term in (2.1). Ting then sought to evaluate α_1 by matching the pressure across III to higher order in R since, as he showed, the $O(R^{-\frac{1}{2}})$ flow in I and II identically satisfies the $O(R^{-\frac{1}{2}})$ pressure balance across III.

To be sure, it might appear reasonable to assume, as Ting did, that the coefficients of all the higher order terms in (2.1) vanish since, for the case considered here, in which the two streams are brought into contact at $x = 0$, the vertical displacement of the dividing streamline is certainly zero at the origin. Such an *a priori* assumption would be incorrect, however, because (2.1), being an asymptotic expansion for $Rx \gg 1$, ceases to apply within the leading-edge region $0 \leqslant Rx \leqslant O(1)$, where viscous forces predominate. Thus, the coefficients in (2.1) must be determined either from a higher order analysis in the boundary-layer region $Rx \gg 1$, or, failing this, by matching (2.1) to an appropriate inner solution that applies near the leading edge. The derivation of the latter, however, presents such apparently insurmountable difficulties that this second approach is rarely, if ever, employed.

One is forced, therefore, to proceed with the higher order analysis in the boundary-layer region, where the absence of a characteristic length scale for the problem requires the solution to have a self-similar form to all orders of approximation. Thus, defining the similarity variable $\zeta = y/x$ for the inviscid regions, we obtain the expansions

$$\Psi_{\mathrm{I}}(x,\zeta) = x\zeta+(x/R)^{\frac{1}{2}}\{f_{11}(\zeta)+(Rx)^{-\frac{1}{2}}f_{12}(\zeta)+\ldots\}, \tag{2.3}$$

$$\Psi_{\mathrm{II}}(x,\zeta) = U_2x\zeta+(x/R)^{\frac{1}{2}}\{f_{21}(\zeta)+(Rx)^{-\frac{1}{2}}f_{22}(\zeta)+\ldots\}, \tag{2.4}$$

$$\Psi_{\mathrm{III}}(x,\eta) = (x/R)^{\frac{1}{2}}\{f_{31}(\eta)+(Rx)^{-\frac{1}{2}}f_{32}(\eta)+\ldots\}, \tag{2.5}$$

together with the associated boundary conditions

$$\Psi_{\mathrm{I}}(x,0) = \Psi_{\mathrm{II}}(x,0) = 0 \quad \text{for} \quad x < 0,$$

$$\Psi_{\mathrm{III}}(x,0) = 0,$$

$$\Psi_{\mathrm{I}}(x,\zeta)|_{\zeta\to0^+} \leftrightarrow \Psi_{\mathrm{III}}(x,\eta)|_{\eta\to\infty},$$

$$\Psi_{\mathrm{II}}(x,\zeta)|_{\zeta\to0^-} \leftrightarrow \Psi_{\mathrm{III}}(x,\eta)|_{\eta\to-\infty},$$

$$\Psi_{\mathrm{I}}(x,\zeta) = x\zeta, \quad \Psi_{\mathrm{II}}(x,\zeta) = U_2x\zeta \quad \text{as} \quad x\to-\infty,$$

where the double-headed arrows imply matching in an asymptotic sense. $f_{31}(\eta)$ in (2.5) corresponds to Lock's (1951) solution and satisfies

$$2f'''_{31}+f_{31}f''_{31} = 0, \quad f_{31}(0) = 0. \quad f'_{31}(\infty) = 1, \quad f'_{31}(-\infty) = U_2. \tag{2.6}$$

For purposes of matching (2.3) and (2.4) as $\zeta \to 0$ with the forms of (2.5) as $\eta \to \pm\infty$ respectively, we require the asymptotic forms of $f_{31}(\eta)$, which, in view of (2.1) and (2.2), become

$$f_{31}(\eta) \sim \begin{cases} \eta - \beta_1 = (Rx)^{\frac{1}{2}}\zeta - (\alpha_1+\beta_1) - \alpha_2(Rx)^{-\frac{1}{2}} + \ldots & \text{as } \eta \to \infty, \\ U_2\eta - \beta_2 = U_2(Rx)^{\frac{1}{2}}\zeta - (U_2\alpha_1+\beta_2) - U_2\alpha_2(Rx)^{-\frac{1}{2}} + \ldots & \text{as } \eta \to -\infty, \end{cases} \tag{2.7}$$

where β_1 and β_2 are positive constants determined from the integration of (2.6). Then, using (2.7) to determine $f_{11}(0)$ and $f_{21}(0)$, we can readily show that

$$\left.\begin{aligned} f_{11}(\zeta) &= -\frac{1}{\sqrt{2}}(\alpha_1+\beta_1)\,[(1+\zeta^2)^{\frac{1}{2}} + \operatorname{sgn} x]^{\frac{1}{2}}, \\ f_{21}(\zeta) &= -\frac{1}{\sqrt{2}}(U_2\alpha_1+\beta_2)\,[(1+\zeta^2)^{\frac{1}{2}} + \operatorname{sgn} x]^{\frac{1}{2}}, \end{aligned}\right\} \tag{2.8}$$

which clearly indicates that the $O(R^{-\frac{1}{2}})$ pressure balance across II is trivially satisfied since $f'_{11}(0) = f'_{21}(0) = 0$.

To resolve the indeterminacy of the solution for $f_{31}(\eta)$, Ting (1959) integrated the y component of the momentum equation across the boundary layer, thereby obtaining an expression for the $O(R^{-1})$ pressure drop. Then, by matching this pressure term at each edge of the boundary layer to that in the adjacent free stream, he derived a compatibility condition for the unknown coefficient α_1. Following this procedure, and using the above expansions, yields

$$(\alpha_1+\beta_1)^2 - (U_2\alpha_1+\beta_2)^2 = 8[f'_{12}(0) - U_2 f'_{22}(0)], \tag{2.9}$$

which corresponds to Ting's result when the right-hand side vanishes. Although Ting argued that $f_{12}(\zeta) \equiv f_{22}(\zeta) \equiv f_{32}(\eta) \equiv 0$, we shall now demonstrate that this is not the case and that, in fact, additional unknown constants appear in (2.9) and preclude the unique evaluation of α_1.

We begin by considering the second-order boundary-layer equation which, through substitution of (2.5) into the x momentum equation in the usual manner, becomes

$$\left.\begin{aligned} 2f'''_{32} + f_{31}f''_{31} + f'_{31}f'_{32} &= 0, \\ \text{with} \qquad f_{32}(0) = f'_{32}(\infty) = f'_{32}(-\infty) &= 0, \end{aligned}\right\} \tag{2.10}$$

since $f'_{11}(0)$ and $f'_{21}(0)$ both vanish. Certainly, $f_{32}(\eta) \equiv 0$ is a solution to (2.10); however, there exists in addition an eigensolution given by (Klemp & Acrivos 1972)

$$f_{32}(\eta) = K[f'_{31}(\eta)/f'_{31}(0) - 1], \tag{2.11}$$

where K is a constant which, as is usually the case with boundary-layer problems of this type, can be determined only by examining the details of the flow in the vicinity of the leading edge. Although the appearance of an eigensolution generally introduces a logarithmic term into the expansion (Van Dyke 1964), this does not occur in the present problem since the eigensolution is the only non-trivial solution of (2.10).

Turning now to the second-order solutions for the inviscid streams, we find that by matching (2.3) and (2.4) to (2.5) in their respective regions of overlap, and using (2.7) and (2.11) as $\eta \to \pm\infty$, we obtain the necessary boundary conditions for the harmonic functions $f_{12}(\zeta)$ and $f_{22}(\zeta)$, which thereby become

$$\left.\begin{aligned} f_{12}(\zeta) &= -\left\{\alpha_2 - K\left[\frac{1}{f'_{31}(0)} - 1\right]\right\}\left[1 - \frac{1}{\pi}\tan^{-1}\zeta\right] \\ f_{22}(\zeta) &= -\left\{U_2\alpha_2 - K\left[\frac{U_2}{f'_{31}(0)} - 1\right]\right\}\frac{1}{\pi}\tan^{-1}\zeta \end{aligned}\right\} 0 \leqslant \tan^{-1}\zeta \leqslant \pi. \qquad (2.12)$$

Consequently, (2.9) reduces to

$$(\alpha_1+\beta_1)^2 - (U_2\alpha_1+\beta_2)^2 = \frac{8}{\pi}\left\{(1+U_2^2)\left[\alpha_2 - \frac{K}{f'_{31}(0)}\right] + (1+U_2)K\right\}. \qquad (2.13)$$

Clearly, since both the higher order displacement of the $\Psi = 0$ streamline and the eigensolution of the second-order boundary-layer equation introduce additional unknown constants in (2.13), α_1 cannot be evaluated from this expression. We conclude, therefore, that if the two uniform streams are indeed semi-infinite in extent then α_1 remains unknown as far as this analysis is concerned. This same conclusion also applies to subsonic compressible flows.

3. Discussion

The analysis presented above demonstrates that a fundamental indeterminacy exists in the solution to the problem of the subsonic mixing of two uniform parallel semi-infinite streams, at least within the framework of classical boundary-layer theory. In spite of this result, however, the problem is not necessarily ill posed, for, although any value of α_1 appears to be consistent with the boundary-layer solution (which, of course, ceases to apply near the point where mixing begins), it might still be possible, as mentioned earlier, to determine α_1 by considering the flow within the leading-edge region, $0 \leqslant |Rx| \leqslant O(1)$, i.e. in the close proximity of the origin. Then, in the absence of any outer boundaries whatsoever, this value of α_1 would continue to apply throughout the flow field.

Of course, in all physical problems the streams are not truly semi-infinite in extent; in fact, regardless of their actual widths, at least one of the bounding surfaces will be located within a distance of order one from the viscous mixing layer after the system has been rendered dimensionless. Thus, to obtain a unique solution for flow in the mixing region between the two uniform subsonic streams, one must abandon the assumption that these are effectively semi-infinite. Then if their finite width is properly taken into account, the $O(R^{-\frac{1}{2}})$ component of the pressure will no longer balance identically across the mixing layer and, consequently, the third boundary condition can be determined from the $O(R^{-\frac{1}{2}})$ analysis and is similar to that presented by Ting (1959) for cases in which one or both streams are supersonic. This solution procedure is quite straightforward and has been applied by Klemp (1971) to a mixing problem involving two uniform streams of finite width.

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